$$\prod_{h=2}^{h=k} b \left(n_h,\ \left\{ \frac{x-1}{n_1 n_2 \ldots n_h} \right\} n_1 n_2 \ldots n_{h-1} + \left\{ \frac{y-1}{\dfrac{n}{n_1 n_2 \ldots n_{h-1}}} \right\} + 1 \right)$$

$$\left(R \frac{y-1}{\dfrac{n}{n_1 n_2 \ldots n_{h-1}}} \right) + 1,\ \left\{ R \frac{x-1}{\dfrac{n_1 n_2 \ldots n_h}{n_1 n_2 \ldots n_{h-1}}} \right\} + 1 \qquad \ldots . (2).$$

Each of the elements $b_{\alpha_g \beta_g}{}^{(n_g,\ i_g)}$ $(\alpha_g,\ \beta_g = 1,\ 2 \ldots n_g)$ occurs in $\dfrac{n}{n_g \ldots . n_k}$ rows and in $\dfrac{n}{n_1 \ldots . n_g}$ columns of A.

If we write $i_g = \lambda_g . n_1 n_1 \ldots n_{g-1} + \mu_g$, where

$$\left\{ \lambda_g = 0,\ 1 \ldots \left(\frac{n}{n_1 \ldots . n_g} - 1 \right) \right\},$$

$$\{ \mu_g = 1,\ 2 \ldots . (n_1,\ n_2 \ldots . n_{g-1}) \},$$

and understand that when $g=1$, $n_1 n_2 \ldots . n_{g-1} = 1$.

Then the element $b_{\alpha_g \beta_g}{}^{(n_g,\ \lambda_g n_1 \ldots . n_{g-1} + \mu_g)}$ occurs in the $(\lambda_g\, n_g + \alpha_g)$th $n_1 n_2 \ldots . n_{g-1}$ rows, and in the $\{(\mu_g - 1) n_g + \beta_g\}$th $\dfrac{n}{n_1 \ldots . n_g}$ columns of A.

4. If all the determinants of the same order n_g, $(g = 1,\ 2 \ldots k)$ are equal the theorem becomes

$$A = \Pi \mid b_{1 n_g}{}^{(n_g)} \mid^{ig},\ ig = n / n_g \qquad \ldots (3).$$

If $n_1 = n_2 = n_3 = \ldots . = n_k = m$, then $n = m^k$ and the theorem takes the form

$$A = \Pi \mid b_{1 m}{}^{(h)} \mid ,\ (h = 1,\ 2 \ldots . k . m^{k-1}) \qquad \ldots (4).$$

If all the determinants are of the same order and equal to each other then the theorem becomes

$$A = \mid b_{1 m} \mid^{k . m^{k-1}} \qquad \ldots (5).$$

After finding the general theorem in Art. 3, the special cases (3) and (5) were first made known to me by Prof. E. H. Moore, who discovered them before knowing of the general theorem, and who immediately made the generalization, using the different notation, on receiving the Muir reference.

December 5, 1898.

THE GEOMETRIC OLD AND NEW.

By DR. GEORGE BRUCE HALSTED.

1. Perhaps the newest geometry in English is by James Howard Gore, professor of mathematics in Columbian University, and published by Longmans, Green & Co., first edition December, 1898, second edition, 1899. Mr. Gore defines the straight in his §6, as follows: "A straight line is the shortest line between two points." But this presumes beforehand the measurement of all lines, while the soundest of geometries, Euclid, will not even attribute length to the simplest of curves, the circle, and our new mathematics knows of lines, real boundaries between two parts of a plane, to which the idea of length is inapplicable. Moreover, before the execution of any measurement there must be a measuring standard; but this is first given by the straight line, is in fact always *a sect*, a definite piece of a straight. Again the existence of a minimum is presumed; which is not evident; and of a single minimum; which is a subtle assumption.

In fact the operation of measuring a geometric magnitude we cannot effect, rigorously speaking, either for curves or curved surfaces. For, however little may be the parts of a curve, they do not cease to be curves, and consequently they cannot be compared with a sect by superposition or congruence; just as parts of a curved surface are not comparable with portions of a plane by superposition or congruence. A paradoxical assumption is in fact necessary, which destroys by itself the primitive idea of measurement, the application of a standard unit.

Thus the evaluation of the length of a curve represents not at all a measurement of the primary kind, and until explicit post-Euclidean assumptions have been made, we cannot even know what is to be meant by one line being shorter than another between the same two points.

2. Mr. Gore defines an angle in §30: "An angle is the difference in direction of two lines that meet." But the word 'direction' can only be given a meaning when the theory of parallels is presupposed. No one has ever been able to say when two different straights have the same direction without using parallelism. Direction, to be understood in any strict sense whatever, presupposes three fundamental geometric ideas, namely, straight line, angle, parallels.

3. In §52 in the last step of an attempted demonstration, we read: "this would give two straight lines joining P and P', *which is impossible.*" On the contrary, in Riemann's double elliptic geometry every two straight lines meet twice.

4. Under Parallel Lines, we read: "§59. Definition. Two straight lines are called Parallel when they lie in the same plane, and cannot meet, *nor approach each other*, however far they may be produced."

Mr. Gore's unfortunate interpolation is thus paraphrased in his next article: "Since parallel lines cannot approach each other, they are everywhere

equally distant from each other." This assumes that a line which is everywhere equidistant from a straight line must be itself straight, but in Bolyai's geometry this line is a curve, the well-known *equidistantial*, while in Lobachevski one of the first theorems is: "The farther parallel lines are prolonged on the side of their parallelism, the more they approach one another."

In §62, in place of a proof for the fundamental theorem: "If two parallels are cut by a transversal the alternate angles are equal," Mr. Gore gives the flat *petitio principii*: "The lines AB and CD, being parallel, have the same direction. The lines EG and GH, being in one and the same straight line, are similarly directed. That is, the angles EGB and GHD have sides with the same direction; therefore the differences of their directions are equal." ! ! !

"§127. A circle is a plane figure"!

"§138. Two circumferences are tangent to each other when they are tangent to a straight line at the same point"! What about two circles with only one point in common, and the straight tangent to one at that point different from the straight tangent to the other at that point?

Page 64, "Up to the present time it has been assumed that any needful line or combination of lines could be drawn, and the question has not arisen as to the possibility of drawing these lines *with accuracy*! ! !

In order to show that any required combination of lines, angles, or parts of lines or angles fulfilled the required conditions, principles were needed long before they could be demonstrated"! Mr. Gore has perhaps never looked into a copy of Euclid. Beyond his three postulates, hypothetical constructions are neither necessary nor admissible.

There is one good purpose that this book may subserve, that is to show how absolutely essential is a knowledge of non-Euclidean geometry.

Austin, Texas, 1900.

EXPRESSION OF RIEMANN'S P FUNCTION AS A DEFINITE INTEGRAL.

By W. E. HEAL.

Consider the differential equation of Riemann's P function, namely,

$$(x-a)(x-b)(x-c)\frac{d^2y}{dx^2}+(x-a)(x-b)(x-c)\left(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}\right.$$

$$\left.+\frac{1-\gamma-\gamma'}{x-c}\right)\frac{dy}{dx}+\left(\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}\right.$$

$$\left.+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right)y=0 \ldots . (1),$$

where α, α', β, β', γ, γ' are real quantities and we have $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$.

Let $V=(x-a)^{\alpha}(x-b)^{\beta}(x-c)^{\gamma}u^{-(\alpha'+\beta'+\gamma')}$

$$(1-u)^{-(\alpha'+\beta+\gamma)}[(a-b)(x-c)+(b-c)(x-a)u]^{-(\alpha+\beta+\gamma)}.$$

Then $y_1=\int_0^1 V du$ is an integral of equation (1).

For substituting this value for y in the equation (1) the left member becomes

$$\int_0^1\Big[(x-a)(x-b)(x-c)\Big\{\frac{(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+1)[(a-b)+(b-c)u]^2}{[(a-b)(x-c)+(b-c)(x-a)u]^2}$$

$$-\frac{(\alpha+\beta+\gamma)\Big(\frac{1+\alpha-\alpha'}{x-a}+\frac{1+\beta-\beta'}{x-b}+\frac{1+\gamma-\gamma'}{x-c}\Big)[(a-b)+(b-c)u]}{[(a-b)(x-c)+(b-c)(x-a)u]}$$

$$+\Big(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\Big)\Big(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\Big)$$

$$+\Big(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\Big)^2-\Big[\frac{\alpha}{(x-a)^2}+\frac{\beta}{(x-b)^2}+\frac{\gamma}{(x-c)^2}\Big]\Big\}$$

$$+\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\Big]Vdu.$$

Since $(x-a)(x-b)(x-c)\Big\{\Big(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\Big)$

$$\Big(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\Big)+\Big(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\Big)^2$$

$$-\Big[\frac{\alpha}{(x-a)^2}+\frac{\beta}{(x-b)^2}+\frac{\gamma}{(x-c)^2}\Big]\Big\}$$

$$+\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}$$

$$=(\alpha+\beta+\gamma)[(\alpha+\beta+\gamma+1)x-(\alpha'+\beta+\gamma)a-(\alpha+\beta'+\gamma)b-(\alpha+\beta+\gamma')c],$$

this result may be written,

$$(\alpha+\beta+\gamma)\int_0^1\Big[(x-a)(x-b)(x-c)\Big\{\frac{(\alpha+\beta+\gamma+1)[(a-b)+(b-c)u]^2}{[(a-b)(x-c)+(b-c)(x-a)u]^2}$$

$$-\frac{\left(\frac{1+\alpha-\alpha'}{x-a}+\frac{1+\beta-\beta'}{x-b}+\frac{1+\gamma-\gamma'}{x-c}\right)[(a-b)+(b-c)u]}{[(a-b)(x-c)+(b-c)(x-a)u]}\Bigg\}$$

$$+[(\alpha+\beta+\gamma+1)x-(\alpha'+\beta+\gamma)a-(\alpha+\beta'+\gamma)b-(\alpha+\beta+\gamma')c]\,V\,du$$

$$=-(\alpha+\beta+\gamma)(a-b)(b-c)(c-a)$$

$$\int_0^1 \frac{\{(\alpha+\beta+\gamma')(a-b)(x-c)+(\alpha'+\beta+\gamma)(b-c)(x-a)u^2}{-[(a-b)(1+\alpha-\alpha')(x-c)+(b-c)(1+\gamma-\gamma')(x-a)]u\}\,V\,du}{[(a-b)(x-c)+(b-c)(x-a)u]^2}$$

$$=-(\alpha+\beta+\gamma)(a-b)(b-c)(c-a)\int_0^1 \frac{d}{du}\left\{\frac{u(1-u)V}{[(a-b)(x-c)+(b-c)(x-a)u]}\right\}$$

$$=-(\alpha+\beta+\gamma)(a-b)(b-c)(c-a)\left\{\frac{u(1-u)V}{[(a-b)(x-c)+(b-c)(x-a)u]}\right\}_0^1$$

which is identically zero if the integral has a meaning. That the integral may not become infinite at the limits we must have

$$1-(\alpha'+\beta'+\gamma)=\alpha+\beta+\gamma'>0,$$
$$1-(\alpha'+\beta+\gamma')=\alpha+\beta'+\gamma>0.$$

It is also clear that

$$y_2=\int_0^{-\infty} V\,du, \quad y_3=\int_1^{+\infty} V\,du,$$

also satisfy the differential equation. For $y=y_2$ we must have

$$\alpha+\beta+\gamma'>0, \quad \alpha'+\beta+\gamma>0.$$

And for $y=y_3$,

$$\alpha+\beta'+\gamma>0, \quad \alpha'+\beta+\gamma>0$$

In equation (1) write

$$a=-1, \quad b=1/\varepsilon, \quad c=+1,$$
$$\alpha=\alpha'=\gamma=\gamma'=0, \quad \beta=-n, \quad \beta'=n+1, \quad \lim\varepsilon=0,$$

and we find after some reductions the differential equation for zonal spherical harmonics, viz:

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{n(n+1)y}{1-x^2} = 0.$$

(Craig's *Linear Differential Equations*, page 192.) In this case

$$y_1 = \int_0^1 \frac{(1-u)^n[(1-x)+(1+x)u]^n du}{u^{n+1}},$$

$$y_2 = \int_0^{-\infty} \frac{(1-u)^n[(1-x)+(1+x)u]^n du}{u^{n+1}},$$

$$y_3 = \int_1^{+\infty} \frac{(1-u)^n[(1-x)+(1+x)u]^n du}{u^{n+1}}.$$

In y_1 we must have n negative and numerically less than unity. For y_3 the values n may have are the same as for y_1. In y_2 we may have n any, negative, real number.

In equation (1) write

$$a=0,\ b=1,\ c=-1,$$
$$\alpha=\tfrac{1}{2}\{(k-1)+\sqrt{[4n(n+k-1)+(k-1)^2]}\},$$
$$\alpha'=\tfrac{1}{2}\{(k-1)-\sqrt{[4n(n+k-1)+(k-1)^2]}\},$$
$$\beta=\gamma=\tfrac{1}{2}\{(2-k)+\sqrt{[4m(m+k-2)+(k-2)^2]}\},$$
$$\beta'=\gamma'=\tfrac{1}{2}\{(2-k)+\sqrt{[4m(m+k-2)+(k-2)^2]}\},$$

and we have the differential equation,

$$\frac{d^2y}{dx^2} + \frac{2x^2+k-2}{x(x^2-1)}\frac{dy}{dx} + \frac{[n(n+k-1)(x^2-1)-m(m+k-2)x^2]y}{x^2(x^2-1)^2} = 0 \quad \ldots\ldots\ldots (3),$$

which transformed by the substitution $x=1/t$ becomes the equation for spherical harmonics of rank k, namely,

$$\frac{d^2y}{dt^2} - \frac{kt}{1-t^2}\frac{dy}{dt} + \frac{[n(n+k-1)(1-t^2)-m(m+k-2)]y}{(1-t^2)^2} = 0 \quad \ldots\ldots\ldots (4),$$

(Craig, page 195.)

The definite integral appears to be too complicated to be of much use except in special cases.

I. Let $k=1$.

$\therefore\ \alpha=n,\ \alpha'=-n,\ \beta=\gamma=\tfrac{1}{2}m,\ \beta'=\gamma'=\tfrac{1}{2}(1-m)$.

Equation (4) becomes for this case,

$$\frac{d^2y}{dt^2} - \frac{t}{1-t^2}\frac{dy}{dt} + \frac{[n^2(1-t^2)-m(m-1)]y}{(1-t^2)^2} = 0 \quad \ldots(5).$$

We have,

$$y_1 = \int_0^1 (1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2n-1)}.[2u-(1+t)]^{-(m+n)}du,$$

$$y_2 = \int_0^{-\infty} (1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2n-1)}.[2u-(1+t)]^{-(m+n)}du,$$

$$y_3 = \int_1^{+\infty} (1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2n-1)}.[2u-(1+t)]^{-(m+n)}du.$$

In all these integrals we have

$$n+\tfrac{1}{2}>0,$$

and in y_2, y_3 we must also have

$$m>n.$$

II, Let $k=2$.

$\therefore\ \alpha=n+1,\ \alpha'=-n,\ \beta=\gamma=\frac{1}{2}m,\ \beta'=\gamma'=-\frac{1}{2}m.$

Equation (4) becomes

$$\frac{d^2y}{dt^2} - \frac{2t}{1-t^2}\frac{dy}{dt} + \frac{[n(n+1)(1-t^2)-m^2]y}{(1-t^2)^2} = 0 \ldots(6),$$

which is (Craig, page 194) the equation of the *associated function*, $P_{n,m}$, of the first kind, of degree n and order m.

We have

$$y_1 = \int_0^1 (1-t^2)^{\frac{1}{2}m}[u(1-u)]^n.[2u-(1+t)]^{-(m+n+1)}du,$$

$$y_2 = \int_0^{-\infty} (1-t^2)^{\frac{1}{2}m}[u(1-u)]^n.[2u-(1+t)]^{-(m+n+1)}du,$$

$$y_3 = \int_1^{+\infty} (1-t^2)^{\frac{1}{2}m}[u(1-u)]^n.[2u-(1+t)]^{-(m+n+1)}du.$$

In these integrals we must have

$$n+1>0,$$

and y_2, y_3 we also have

$$m>n.$$

If in (6) we put $n=\mu-\frac{1}{2}$ we have,

$$\frac{d^2y}{dt^2}-\frac{2t}{1-t^2}\frac{dy}{dt}+\frac{[(\mu^2-\frac{1}{4})(1-t^2)-m^2]y}{(1-t^2)^2}=0\ldots(7),$$

which is (Craig, page 194) the differential equation for Hicks' Toroidal functions.

We have for this case

$$y_1=\int_0^1(1-t^2)^{\frac{1}{4}m}[u(1-u)]^{\frac{1}{4}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{4}(2m+2\mu+1)}\,du,$$

$$y_2=\int_0^{-\infty}(1-t^2)^{\frac{1}{4}m}[u(1-u)]^{\frac{1}{4}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{4}(2m+2\mu+1)}\,du,$$

$$y_3=\int_1^{+\infty}(1-t^2)^{\frac{1}{4}m}[u(1-u)]^{\frac{1}{4}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{4}(2m+2\mu+1)}\,du,$$

where $\mu+\frac{1}{2}>0$, and in y_2, y_3 $m>\mu-\frac{1}{2}$.

FORECASTING THE CENSUS RETURNS.

By JAMES S. STEVENS. Professor of Physics, The University of Maine, Orono, Maine.

Now that the government of the United States is about to take another census, we occasionally see in the newspapers forecasts of the population. Sometimes these forecasts are mere guesses, but there is a method by which one can make these estimates scientifically, and if they fail to come out right it is the fault of the people rather than the method.

All physical laws may be divided into two classes—rational and empirical. The free fall of a body, the swinging of a pendulum, and most of the laws of heat and electricity illustrate the first class. If a body falls one space the first second it will fall three the second and five the third. These laws may easily be embodied into formulae which contain no arbitrary constants. On the other hand, such problems as the relation between temperature and depth below the

surface, and the gravitational acceleration over the earth's surface follow no fixed law. In order to express them at all, special formulae must be devised which contain constants whose values are to be determined.

It is obvious that the law of increase of population of a country belongs to this latter class. To construct an empirical formula a curve is plotted showing the relation between the various decades and their corresponding population. When this is drawn, unless it is too irregular, a mathematician can locate it among the curves with which he is familiar, and the equation of the curve is the empirical formula required. The census curve for the United States turns out to be a parabola whose equation is $p=S+Tx+Ux^2$ in which p is the population, x the number of the decade, and S, T, and U are unknown constants.

To determine these constants we first write down the population for the various decades. (These figures are from the World Almanac.)

1790	—	3.6
1800	—	5.3
1810	—	7.2
1820	—	9.6
1830	—	12.9
1840	—	17.1
1850	—	23.2
1860	—	31.4
1870	—	38.6
1880	—	50.2
1890	—	62.6

The population is expressed in millions and tenths of a million. Substituting these values in our formula we have:

$$
\begin{aligned}
3.6 &= S+ T+ U\\
5.3 &= S+ 0+ 0\\
7.2 &= S+ T+ U\\
9.6 &= S+2T+ 4U\\
12.9 &= S+3T+ 9U\\
17.1 &= S+4T+16U\\
23.2 &= S+5T+25U\\
31.4 &= S+6T+36U\\
38.6 &= S+7T+49U\\
50.2 &= S+8T+64U\\
62.6 &= S+9T+81U
\end{aligned}
$$

These are called "observation equations." If we multiply each one in turn by the coefficients of S, T, and U respectively, and add the results, we have what are called "normal equations." They are as follows:

$$
\begin{array}{rrrrr}
-3.6 = & -S + & T - & U \\
7.2 = & S + & T + & U \\
19.2 = & 2S + & 4T + & 8U \\
38.7 = & 3S + & 9T + & 27U \\
68.4 = & 4S + & 16T + & 64U \\
116.0 = & 5S + & 25T + & 125U \\
188.4 = & 6S + & 36T + & 216U \\
270.2 = & 7S + & 49T + & 343U \\
401.6 = & 8S + & 64T + & 512U \\
563.4 = & 9S + & 81T + & 728U \\
\hline
1669.5 = & 44S + & 286T + & 2023U
\end{array}
$$

$$
\begin{array}{rrrrr}
3.6 = & S - & T + & U \\
7.2 = & S + & T + & U \\
38.4 = & 4S + & 8T + & 16U \\
116.1 = & 9S + & 27T + & 81U \\
273.6 = & 16S + & 64T + & 256U \\
580.0 = & 25S + & 125T + & 625U \\
1130.4 = & 36S + & 216T + & 1296U \\
1891.4 = & 49S + & 343T + & 2401U \\
3212\ 8 = & 64S + & 512T + & 4096U \\
5070.6 = & 81S + & 729T + & 6561U \\
\hline
12324.1 = & 286S + & 2026T + & 15334\mathrm{U}
\end{array}
$$

When we multiply through by the coefficients of S (unity) the observation equations undergo no change and their sum is :

$$261\ 7 = 11S + 44T + 286\mathrm{U}.$$

We have now three equations with three unknown quantities which we may solve in any manner we choose. The values obtained are :

$$S = 6.08$$
$$T = 0.690$$
$$\mathrm{U} = 0.622$$

If we substitute these in the formula we get :

$p = 6.08 + 6.9 + 62.2$, or
$p = 75.2$ millions, which is the forecast for 1900.

If we neglect the curves of 1890 the formula becomes :

$$p = 4.97 + 0.873x + 0.581x^2.$$

If we use only those from 1820 to 1880 inclusive we get :

$$p=7.29-0.28x+0.689x^2.$$

For the year 1894 the first of the formula gave 64.5 and the second 65.5 ; while the population reported in the World Almanac was 66.7.

The second formula gives for 1900 a population of 73.4, while the first formula gives a result somewhat lower. It appears then, that the population is increasing more rapidly than the parabolic curve indicates, and that if anything our forecast of 73 4 is somewhat low. It must be understood of course that the above formulae are strictly anti-expansion and make no allowance for our new possessions.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from the February Number.]

CHAPTER VI.

GEOMETRICAL ADDITION AND SUBTRACTION.

73. The general theory of the *Ausdehnungslehre* may be applied in such diverse sciences as geometry, mechanics, and logic. We proceed in this and the following chapter to apply it in geometry.

74. The concepts dealt with in geometry are the *point*, *line*, *surface*, and *solid*, which may or may not be fixed in position. For the sake of distinction a line whose length and direction are fixed but not its position is called a *vector*. (3). A portion of a plane whose direction and extent are fixed but not the position of the plane is called by analogy a *plane vector*.

75. As an introduction to the *Ausdehnungslehre* the addition and subtraction of vectors was treated in Chapter I. It is evident from what was given in that chapter that plane vectors may be added and subtracted in the same way as line vectors. One gets the parts whose sum is a given plane vector by projecting the given plane vector on the coördinate planes.

As we have already treated of the addition and subtraction of vectors, we proceed to apply the laws of addition and subtraction to points.

76. We will define a point as an infinitesimal portion of a line and denote it by p. When the point has position we will denote it by $pO\rho$, in which p denotes the point at the extremity of the radius vector ρ from the origin, O. Evidently a line, or plane, or solid may be located by means of a radius vector in the same way. (See 168.)

Grassmann defines a point as that which has position and uses a single letter as A to denote it. In what follows if the ρ's be cut out of the formulas and the p's be given the subscripts of the ρ's, Grassmann's expressions will result.

The reasons for using the complex symbol $p\rho$, instead of the simpler A, are: (1) Because the concept is complex and therefore for clearness should be represented by a complex symbol. (2) Since position is relative, for the proper representation of positions an origin is needed. (3) Because this notation shows plainly the relation which exists between point and vector analysis.

77. What we will call *unit* points all have the same (infinitesimal) unit length. This length as also that of the radius vector may be multiplied by any scalar, m. Thus $mp\rho$ denotes the point whose length or "weight," as it is called, is mp held in position by ρ, while $p(m\rho)$ denotes p held in position by $m\rho$, *i. e.*, m times the length of ρ.

78. The difference between two unit points, since they can differ only in position, is a certain distance in a certain direction, *i. e.*, is a vector. (See 3.)

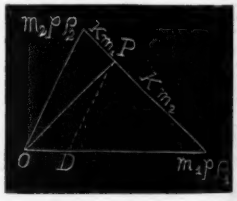
Thus $p\rho_2 - p\rho_1 = \rho_2 - \rho_1 = \varepsilon$. (4).

Similarly, $mp\rho_2 - mp\rho_1 = m\varepsilon$.

79. We next seek to find the sum of two or more points. What this sum is remains to be determined. Grassmann gives an investigation to show that the sum of two unit points is a point on the line joining them. We abridge this proof as follows: He begins by postulating (1) That whatever is true of one set of points is true of any congruent system wherever situated; (2) That the fundamental laws of addition and subtraction (14) hold. Then he assumes that the sum of two points is some point.

Let, in the figure, $p\rho_1$ and $p\rho_2$ be any two unit points whose sum is sought. Suppose $p\rho_1 + p\rho_2 = p\rho_x$. Then revolving the whole figure in the plane of the paper through 180°, $p\rho_1$ coincides with $p\rho_2$, $p\rho_x$ with $p\rho_y$, and $p\rho_2$ with $p\rho_1$. Thus we get $p\rho_2 + p\rho_1 = p\rho_y$. But by 14, $p\rho_2 + p\rho_1 = p\rho_1 + p\rho_2$. Then $p\rho_x = p\rho_y$. This can only happen when they both coincide with the midpoint of the straight line joining the two given points.

80. Mechanics gives us a simpler and more general interpretation for the sum of two or more points. Let us regard the points as parallel (infinitesimal) forces whose magnitudes are represented by the weights of the points. The law for the addition of parallel forces gives a simple and consistent result. Thus



$$m_1 p\rho_2 + m_2 p\rho_2 = (m_1 + m_2)p\left(\frac{m_1\rho_1 + m_2\rho_2}{m_1 + m_2}\right),$$

i. e., the sum of the two weighted points is a point on the line joining them whose

weight is the sum of the weights of the two points and the extremity of whose radius vector divides the line joining the two given points into segments inversely proportional to the weights of these points.

That $\frac{m_1\rho_1+m_2\rho_2}{m_1+m_2}$ is the vector to the point described is evident from elementary geometry. Thus regarding the two radii vectores going out from O as axes, it is easy to show by similar triangles that

$$OD=\frac{m_1}{m_1+m_2}\rho_1, \text{ and } DP=\frac{m_2}{m_1+m_2}\rho_2.$$

NOTE.—The letter k which appears on the figure above is a factor chosen such that km_1 and km_2 equal the segments designated by them.

81. *Generalizing the result of the last article, we have*

$$\Sigma m_1 p\rho_1=\Sigma m_1 . p\left(\frac{\Sigma m_1\rho_1}{\Sigma m_1}\right).$$

82. When $\Sigma m_1=0$ in the preceding result, the weight of the sum point is zero and the radius vector is infinite in length. To interpret this, we get the sum of all the points except one and then add this partial sum to the remaining point. In this way we obtain an expression similar to that of 78 where the result is a *vector*.

Hence $\Sigma m_1 p\rho_1$ is a VECTOR when $\Sigma m_1=0$, and a POINT when Σm_1 is not equal to 0.

Thus a point at infinity (of zero weight) is equivalent to a vector.

83. Using the formula of 80 and putting

$$(m_1+m_1)=-m_3 \text{ and } \frac{m_1\rho_1+m_2\rho_2}{m_1+m_2}=\rho_3,$$

we see that $m_1p\rho_1+m_2p\rho_2+m_3p\rho_3=0$, and $m_1+m_2+m_3=0$, *are the conditions that the three points $p\rho_1$, $p\rho_2$, $p\rho_3$ shall be collinear and the the vectors ρ_1, ρ_2, ρ_3, coplanar.*

84. For space of three dimensions we have (82)

$$m_1p\rho_1+m_2p\rho_2+m_3p\rho_3+m_4p\rho_4=0, \text{ and } m_1+m_2+m_3+m_4=0,$$

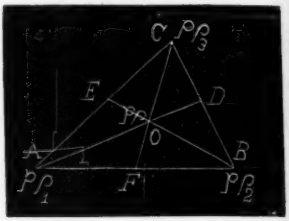
for the conditions that the four points $p\rho_1$, $p\rho_2$, $p\rho_3$, $p\rho_4$ shall be coplanar.

85. *From the equations of* 80, 83, 84 *we see that two points are independent, three or more collinear points are dependent* (10), *three non-collinear points are independent, four or more coplaner points are dependent, four non-coplaner points are independent, and any five or more points are dependent in solid space.*

86. The calculus of this chapter is evidently adapted to dealing with theorems concerning the collinearity of points in geometry, and the center of parallel forces in mechanics.

87. We conclude this chapter with an example. *Required to find whether the three medians of a triangle meet in a point.*

Let the vertices of a triangle ABC be located by the unit points $p\rho_1$, $p\rho_2$, $p\rho_3$, and D, E, F be the mid-points of the sides. We have then



$$D=\frac{p\rho_2+p\rho_3}{2},\quad E=\frac{p\rho_1+p\rho_3}{2}.$$

If $p\rho$ denote a unit point at O the intersection of AD and BE, and x, y, x', and y' arbitrary scalars, we may write

$$p\rho=xp\rho_1+y\frac{p\rho_2+p\rho_3}{2}=x'p\rho_2+y'\frac{p\rho_1+p\rho_3}{2}.$$

Then (20), $x=\frac{1}{2}y'$, $y=y'$; whence $x=\frac{1}{2}y$. But $x+y=1$ (77). Then $x=\frac{1}{3}$, $y=\frac{2}{3}$. Hence

$$p\rho=\tfrac{1}{3}p\rho_1+\tfrac{2}{3}\left(\frac{p\rho_2+p\rho_3}{2}\right)=\tfrac{1}{3}p\rho_1+\tfrac{1}{3}p\rho_2+\tfrac{1}{3}p\rho_3.$$

By symmetry we see that the intersection of AD and CF must be the same point. Or, supposing O to be the intersection of BE and CF, we may test A, O, and D for collinearity directly.

$$\begin{matrix} A & & O & & D \\ \tfrac{1}{3}p\rho_1- & | & (\tfrac{1}{3}p\rho_1+\tfrac{1}{3}p\rho_2+\tfrac{1}{3}p\rho_3)+\tfrac{2}{3}(\tfrac{1}{2}(p\rho_2+p\rho_3)) & \equiv & 0. \quad (83). \end{matrix}$$

It is evident that the above equations can be interpreted as equations of ordinary vector analysis by dropping the p's. In this way is shown the relation existing between point and vector analysis.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

127. Proposed by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

A man borrows \$1000 of a Building and Loan Association, and at the same time subscribes for 10 \$100-shares of stock. A membership fee of \$1 per share is charged. At the beginning of each month an installment of \$1 per share is paid, also 5% interest and 5% premium on the \$1000. The stock matures in 75 months and the debt is cancelled. What rate of interest does he pay per annum?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

\$10 membership fee, \$10 installment, and \$8.33⅓ interest and premium per month=\$28.33⅓ amount paid down.

\$1000−\$28.33⅓=\$971.66⅔ actual amount received after deducting amounts paid at time of borrowing. On this amount, \$18.33⅓ per month is paid for 74 months.

$\therefore 18.33\tfrac{1}{3} = \dfrac{971.66\tfrac{2}{3}r(1+r)^{74}}{(1+r)^{74}-1}$, or $18\tfrac{1}{3}(1+r)^{74} - 18\tfrac{1}{3} = 971\tfrac{2}{3}r(1+r)^{74}$.

$(1+r)^{74} - 1 = 53r(1+r)^{74}$.

$\therefore (1-53r)(1+r)^{74} = 1$.

$\therefore \log(1-53r) + 74\log(1+r) = 0$.

$\therefore r = .0137$, and $12r = .1644 = 16.44\%$ per annum.

128. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

At what time is the figure 7, on the face of a clock, midway between the hour and minute hands?

Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; D. G. DORRANCE, Jr., Camden, N. Y.; G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; and the PROPOSER.

Put $7 = a$ = the given figure on face of clock.

Let x = distance the hour hand travels after a o'clock.

Then $5a - x$ = distance the minute hand travels to fulfill the condition between a and $a+1$ o'clock.

$\therefore$ As the minute hand goes 12 times as fast as the hour hand, $5a - x = 12x$, and $x = \dfrac{5a}{13}$.

This is the *first* position of the hour hand after a o'clock. For, it will be observed, there are thirteen different positions in all: one after each hour and one at $2a$ o'clock.

For the *second* position, which is between $a+1$ and $a+2$ o'clock, $60+5a-x$ = distance the minute hand has to travel. Whence, $60+5a-x = 12x$, and $x = \dfrac{60+5a}{13}$.

Now let n represent these 13 positions of the hour hand. Then $x = \dfrac{60(n-1)+5a}{13}$ = the nth position of the hour hand after a o'clock.

The time of day for the different positions is, *before* $2a$ o'clock, $5a - \dfrac{60(n-1)+5a}{13}$, or $\dfrac{60(a-n+1)}{13}$ minutes *past* $a+n-1$ o'clock; and, *after* $2a$ o'clock, $\dfrac{60(n-1)+5a}{13} - 5a$, or $\dfrac{60(n-1-a)}{13}$ minutes *to* $a+n-1$ o'clock.

Substituting 7 for a, and 1, 2, 3 . . . . 13, consecutively, for n, we obtain the following thirteen times of day when 7 is midway between the hour and minute hands: $32\frac{4}{13}$ minutes past 7, $27\frac{9}{13}$ minutes past 8, $23\frac{1}{13}$ minutes past 9,

$18\frac{6}{13}$ minutes past 10, $13\frac{1}{13}$ minutes past 11, $9\frac{3}{13}$ minutes past 12, $4\frac{8}{13}$ minutes past 1, 2 o'clock, $4\frac{8}{13}$ minutes to 3, $9\frac{3}{13}$ minutes to 4, $13\frac{1}{13}$ minutes to 5, $18\frac{6}{13}$ minutes to 6, and $23\frac{1}{13}$ minutes to 7.

Also solved by *JOSIAH H. DRUMMOND, P. S. BERG, ELMER SCHUYLER, H. C. WHITAKER,* and *J. SCHEFFER.*

129. Proposed by J. W. DAPPERT, Civil Engineer and Surveyor, Taylorville, Ill.

"A Minion, agile, in stature small
Panting came to great Diana's Hall,
Bearing a marble globe upon his shoulders,
Measuring one inch in its diameters.
He rolled it to the northeast corner of the Hall,
Left touching the northern and eastern walls;
Then following came three demi-gods in white,
Each bearing a globe of lustrous metal bright;
One of iron, copper one, and one of silver;
And they placed them in the order given,
Touching each the other, and at the same time,
Touching each the side walls, in a direct line,
The iron touching the marble, and its other side
Resting against the silver, in its glory and pride,—
All resting upon the oaken floor; and then
With heavy tread, and puff, and roar, Atlas came
Bearing a huge golden sphere, that filled the Hall,
Touching the four sides, floor and ceiling, and all
Radiant with beauty, resting against the silvery ball,
Making the globe's diameters in the room diagonal."

"Tell me, all ye who mathematics know:
What size the copper sphere, and oh!
How large the iron globe? How great
The golden globe; immaculate?
The silver sphere, how great? What size?
And if presented as a prize,
What value do you hold
Would be the sphere of gold?"

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

A cube, in form, is great Diana's Hall:
The massive sphere of gold, admired by all,
With ceiling, sides and floor in contact is.
Within the northeast corner's boundaries
Are found the other spheres, in number four,
Each tangent to the corner's sides and floor;
Besides, the sizes of the sphere are such
That each one with its neighbor is in touch.
The order, as they to the corner rolled,
Was marble, iron, copper, silver, gold.
The hall's dimensions, length and breadth and height,
And golden sphere's diameter are quite
The same in measurement; let this be b.
The hall's diagonal, which we'll name c,
Is quickly found *b times square root of 3.* [$b\sqrt{3}$]
The points of contact of the spheres we find
Are in the hall's diagonal confined;
So, too, the centers of the spheres are there.
If hall with golden sphere we now compare,
Outside the sphere are equal ends of c;
And, known as d, each end is found to be
Square root of 3 less one times half of b. [$\frac{1}{2}b(\sqrt{3}-1)$]
The ratio of diameters to find
Of tangent spheres, must next be borne in mind.
Take a as silver sphere's diameter;

Then *d* is found to equal, we aver,
Square root of 3 plus one times half of a. [½a(√3+1)]
Equating *d*'s two different values, weigh
Results, and see *square root of 3 plus 2.* (√3+2]
Of *b* to *a* the ratio, come to view.
This ratio known as *t*; then find it true
That of diameters of any two
Contiguous spheres within the corner's space
The ratio is the self-same *t*, a case
Of beauty mathematical. We now
From demonstrations rest the knitted brow,
And the diameters of all the spheres
Announce. Within the problem it appears
That the diameter of marble sphere
Is just *one inch*. From ratio *t* 't is clear
That iron sphere's diameter is *t*;
Then that of copper sphere *t squared* must be;
Whence that of silver sphere is *cube of t*;
And the diameter of golden sphere
As *t involved to fourth power* must appear.
Three hundred seventy-three as hundredths is
The value of the ratio *t*; and this [*t*=3.73+]
To powers second, third and fourth involved,
Yields as results, approximately solved, [13.92+]
Nine tenths to thirteen added, fifty-two, [51.98+]
And ninety-seven multiplied by two. [193.99+]

The golden sphere is wondrous as to size;
And in regard to value as a prize,
Not all our country's golden output coined
In her existing years, together joined,
Could purchase its great worth. In numbers round,
Eight hundred four of millions will be found
The dollars that in purest gold abound
In this great sphere. Besides it would confound
The efforts of the mind that should aspire
To measure the extent of thinnest wire
To which there might be drawn this golden mass.
From Earth to Sun this slender thread could pass,
Return again to Earth, and eight times more
The circuit make, and still have end galore
To stretch from Earth to Moon strands eighteen score.

ALGEBRA.

104. Prize Problem. $2.50 for the best solution.

Compute to three decimal places each of the roots of the equation $x^2+y=2$, $x+y^2=6$.

I. Solution by AGNES E. SCHEFFER, Hagerstown, Md.

From the first of these equations we have $y=2-x^2$, and substituting this in the second, we have $x^4-4x^2+x-2=0$.

Factoring we have $x^2(x^2-4)+(x-2)=0$, or $(x-2)(x^3+2x^2+1)=0$.

$\therefore$ $x-2=0$, and $x^3+2x^2+1=0$.

From the former we obtain $x=2$ and then its simultaneous value $y=-2$, obtained from the first of the original equations.

By solving the cubic equation $x^3+2x^2+1=0$, we obtain three more roots for x.

Putting $x=z-\frac{2}{3}$, we get $z^3-\frac{1}{3}z+\frac{1}{2}\frac{3}{7}=0$, and employing Cardon's formula we have

$$y=\sqrt[3]{[-\tfrac{1}{2}\tfrac{3}{7}+\tfrac{1}{6}\tfrac{1}{4}\sqrt{(1593)}]}+\sqrt[3]{[-\tfrac{1}{2}\tfrac{3}{7}-\tfrac{1}{6}\tfrac{1}{4}\sqrt{(1593)}]}, \text{ or}$$

$$z=\tfrac{1}{3}\{\sqrt[3]{[-\tfrac{4}{2}^3+\tfrac{1}{2}\sqrt{1}(1593)]}+\sqrt[3]{[-\tfrac{4}{2}^3-\tfrac{1}{2}\sqrt{1}(1593)]}\}, \text{ and}$$

$$x=-\tfrac{2}{3}+\tfrac{1}{3}\{\sqrt[3]{[-\tfrac{4}{2}^3+\tfrac{1}{2}\sqrt{1}(1593)]}+\sqrt[3]{[-\tfrac{4}{2}^3-\tfrac{1}{2}\sqrt{1}(1593)]}\}=-2.205569.$$

Dividing x^3+2x^2+1 by $x+2.205569$ we get $x^2-.205569x+.4533966$, and now the equation $x^2-.205569x+.4533966=0$ furnishes us the other two roots of x, viz: $x=.102784\pm.665456\sqrt{-1}$.

The simultaneous values of y we obtain from the equation $x^2+y=2$, viz: $y=2-x^2$.

Thus, we find the following four sets of the simultaneous values of x and y:

$x=2$	$x=-2.205569$	$x=.102784\pm.665456\sqrt{-1}$
$y=-2$	$y=-2.864534$	$y=2.432267\mp.136796\sqrt{-1}$

II. Solution by J. W. YOUNG, Fellow and Assistant in Mathemetics, Ohio State University, Columbus, O.

I. Solve (1) for y, substitute in (2) and obtain

$$x^4-4x^2+x-2=0.$$

In the application of Horner's method or by inspection we see that $2=x_1$ is a root. By dividing out this root we obtain for the equation giving the remaining roots

$$x^3+2x^2+1=0. \ldots (3).$$

This equation has a pair of complex roots. Denoting the real root by x_2, and applying Horner's process, we find

$$x_2=-2.2055+.$$

Denoting the other roots x_3, x_4 by $\alpha\pm i\beta$ and observing that the sum of the roots of (3) equals -2, we have

$$2\alpha-2.2055=-2.$$

Whence $\alpha=0.1027+$.

Similarly, the product of the roots,

$$x_2(\alpha^2+\beta^2)=-1.$$

Whence $\beta=0.6654+$.

Hence the complex roots are x_3, $x_4=0.1027\pm i0.6654+$.

Collecting results and calculating the corresponding values of y, we have, as a complete solution of the original system.

$$x_1 = 2. \qquad y_1 = -2.$$

$$x_2 = -2.2055 \qquad y_2 = -2.8642$$

$$\left.\begin{matrix} x_3 \\ x_4 \end{matrix}\right\} = 0.1027 \pm i0.6654 \qquad \left.\begin{matrix} y_3 \\ y_4 \end{matrix}\right\} = 2.4323 \mp i0.1367$$

II. Since the cubic giving the incommensurable roots has a pair of complex roots, Cardan's solution may be applied. In the cubic (3) put $x = z - \frac{2}{3}$, and obtain

$$z^3 - \tfrac{1}{9}^2 z + \tfrac{4}{2}\tfrac{3}{7} = 0 \ldots . (4).$$

Put $z = \sqrt[3]{p} + \sqrt[3]{q}$, and cube. Then

$$z^3 - 3\sqrt[3]{p}\sqrt[3]{q}z - (p+q) = 0 \ldots . (5).$$

Comparing coefficients in (4) and (5), we have

$$p+q = -\tfrac{4}{2}\tfrac{3}{7}, \quad pq = \tfrac{6}{7}\tfrac{4}{2}\tfrac{1}{9}.$$

Whence $p = -0.0572$ and $\sqrt[3]{p} = -0.3853$, $q = -1.5354$ and $\sqrt[3]{q} = -1.1536$.

Hence the real root of $(4) = \sqrt[3]{p} + \sqrt[3]{q} = -1.5389$ and $x_2 = z - \frac{2}{3} = -1.5389 - 0.7666 = -2.2055$ as before.

The complex roots are given by $w\sqrt[3]{p} + w^2\sqrt[3]{q}$ and $w^2\sqrt[3]{p} + w\sqrt[3]{q}$, where $w = \frac{1}{2}(-1 \pm i\sqrt{3})$. Computing these roots by the above formulae, we obtain as before

$$x_3, \; x_4 = 0.1027 \pm i0.6654.$$

III. In equation (3) substitute $x = \alpha + i\beta$, and obtain, after separating real and imaginary parts

$$\alpha^3 + 2\alpha^2 + 1 - \beta^2(\beta\alpha + 2) = 0 \ldots . (6),$$

$$\beta(\beta^2 - 3\alpha^2 - 4\alpha) = 0 \ldots . (7).$$

From (7), $\beta = 0$,

$$\beta^2 = 3\alpha^2 - 4\alpha \ldots . (8).$$

For $\beta = 0$ (x real) we have then $\alpha^3 + 2\alpha^2 + 1 = 0$ (from (6)) giving the real root, $x_2 = -2.2055$.

For $\beta^2 = 3\alpha^2 - 4\alpha$, we have from (6),

$$8\alpha^3 + 16\alpha^2 + 8\alpha - 1 = 0.$$

Solving this we obtain $\alpha = 0.1027$ as before. And from (8) $\beta = 0.6654$.

IV. We may write (1) and (2) in the form

$$x^2 - 4 = -(2+y)$$

$$x - 2 = 4 - y^2.$$

From which immediately $x=2$, $y=-2$.

Dividing we have $x+2=-\frac{1}{2-y}$, or $y=\frac{2x+5}{x+2}$.

Substitute in (1), and obtain $x^3+2x^2+1=0$. Solve this by any of the above methods.

V. Add (1) and (2) and complete squares. Then

$$x^2+x+\tfrac{1}{4}+y^2+y+\tfrac{1}{4}=8+\tfrac{2}{4} \text{ or } (x+\tfrac{1}{2})^2+(y+\tfrac{1}{2})^2=\tfrac{2}{4}^5+\tfrac{2}{4}.$$

Whence, from the four possible corresponding values of x and y, we may pick out one set which satisfies the original system, namely $x=2$, $y=-2$.

Also solved by *GEO. R. BERRY.*

NOTE. The donor of this prize has acted as judge on the merits of the several solutions, and his decision is that the two published solutions are of equal merit. In accordance with this decision, the prize money has been equally divided between Miss Scheffer and Mr. Young. We might say that there has been only one solution sent in to the prize problem in Mechanics. This solution is defective. The problem is, therefore, open to all our contributors for solution. EDITOR F.

105. Proposed by CHARLES E. MYERS, Canton, O.

Solve for x the following: $a\log(a^x)=m\log(m)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; J. K. ELLWOOD, A. M., Colfax School, Pittsburg, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.; W. F. SHAW, Austin, Tex.; and ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

$a\log(a^x)=m\log m$ may be written $a^{x\log a}=m^{\log m}$.

$\therefore x^2(\log a)^2=(\log m)^2$.

$\therefore x=\pm(\log m/\log a)$.

GEOMETRY.

130. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

If the points x, y, z divide the strokes $c-b$, $a-c$, $b-a$, in the same ratio r, and the triangles x, y, z and a, b, c are similar, either $r=1$ or both triangles are equilateral. [From Harkness and Morley's *Introduction to the Theory of Functions*, page 26].

Solution by the PROPOSER.

Let x, y and z denote the points, dividing $c-b$, $a-c$, and $b-a$, respectively, in the given ratio r.

Then $x=\frac{c+br}{1+r}$, $y=\frac{a+cr}{1+r}$, and $z=\frac{b+ar}{1+r}$.

The condition that a, b, c, and x, y, z form similar triangles is

$$\begin{vmatrix} a & x & 1 \\ b & y & 1 \\ c & z & 1 \end{vmatrix}=0 \ldots . (1).$$

Substituting in this equation, the values of x, y, and z, and reducing, we obtain

$$r(bc+ac+ab-a^2-b^2-c^2)=bc+ac+ab-a^2-b^2-c^2.$$

From this equation we find that $r=1$, unless

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix} =0 \ldots . (2).$$

But (2) is the condition that the triangle a, b, c is equilateral. Therefore, either $r=1$, or else both triangles are equilateral.

Also solved by *G. B. M. ZERR*, *J. W. YOUNG*, and *FRANK A. GRIFFIN*.

131. Proposed by J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.

Prove that $\lambda+\mu\omega+\nu\omega^2$, where λ, μ, ν are integers whose sum is ± 1, represents the points of a quilt formed by regular hexagons. ω=primitive cube root of unity. [From Harkness and Morley's *Introduction to Theory of Functions*.]

Solution by the PROPOSER.

$$\omega=-\tfrac{1}{2}+i(\tfrac{1}{2}\sqrt{3}),\ \omega=-\tfrac{1}{2}-i(\tfrac{1}{2}\sqrt{3})\ (i^2=-1).$$

Then $\lambda+\mu\omega+\nu\omega^2=\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu+i(\mu-\nu)\sin\frac{1}{3}\pi$. Taking rectangular coördinates this quantity represents the points (x, y), when

$$\left.\begin{aligned} x&=\lambda-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu \\ y&=(\mu-\nu)\sin\tfrac{1}{3}\pi \end{aligned}\right\} \begin{matrix} \lambda+\mu+\nu=\pm 1 \\ (\lambda,\ \mu,\ \nu \text{ are integers}) \end{matrix}$$

$y=n\sin\frac{1}{3}\pi$, when $n=(\mu-\nu)$=any integer.
Then $\mu=n+\nu$.
Substituting in x and in $\lambda+\mu+\nu=\pm 1$, we obtain

$$\left.\begin{aligned} 2\lambda-2\nu-n&=2x \\ \lambda+2\nu+n&=\pm 1 \end{aligned}\right\} \ldots . (1).$$

$\therefore\ x=\frac{1}{2}(3\lambda\pm 1)$.

The points required are, then, those whose coördinates are $[\frac{1}{2}(3\lambda\pm 1,\ n\sin\frac{1}{3}\pi]$, where λ, n are any integers with the one restriction that when n is *odd*, λ is even, and *vice versa*. This restriction is evident, since (1) shows that $\lambda+n$ must be *odd*.

We have then following values of x and y:

A.

For $n=0, 2, 4, 6$, etc.

$\lambda=$	1	3	5	7	9	etc.
$x=$	1 2	4 5	7 8	10 11	13 14	

For $n=1, 3, 5, 7$, etc.

$\lambda=$	0	2	4	6	8	
$x=$	$-\frac{1}{2}$ $\frac{1}{2}$	$\frac{5}{2}$ $\frac{7}{2}$	$\frac{11}{2}$ $\frac{13}{2}$	$\frac{17}{2}$ $\frac{19}{2}$	$\frac{23}{2}$ $\frac{25}{2}$	

$y=n\sin\frac{1}{2}\pi.$

Similarly for negative values of n, λ.

MECHANICS.

96. Proposed by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Two particles, subject to their mutual attraction and that of a fixed center, move in a plane containing the center. Find the motion under the law of the inverse square.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Take the center of force as origin. Let m, m_1, m_2 be the masses of the center of force and particles, respectively. r, r_1, ρ the distances of the particles from the center of force and from each other, respectively. (x, y), (x', y') the coördinates of the particles. The differential equations of motion of the two particles relative to the center of force are

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{m+m_1}{r^3}x+\frac{x'-x}{\rho^3}m_2-\frac{m_2x'}{r_1^3}\\ \frac{d^2y}{dt^2} &= -\frac{m+m_1}{r^3}y+\frac{y'-y}{\rho^3}m_2-\frac{m_2y'}{r_1^3}\end{aligned}\right\}\ldots.(1).$$

$$\left.\begin{aligned}\frac{d^2x'}{dt^2} &= -\frac{m+m_2}{r_1^3}x'+\frac{x-x'}{\rho^3}m_1-\frac{m_1x}{r^3}\\ \frac{d^2y}{dt^2} &= -\frac{m+m_2}{r_1^3}y'+\frac{y-y'}{\rho^3}m_1-\frac{m_1y}{r^3}\end{aligned}\right\}\ldots(2).$$

Where $\rho=\sqrt{[(x'-x)^2+(y'-y)^2]}$.

Multiply (1) by $2m_1\frac{dx}{dt}-2m_1\frac{m_1\frac{dx}{dt}+m_2\frac{dx'}{dt}}{m+m_1+m_2}$.

$$2m_1\frac{dy}{dt}-2m_1\frac{m_1\frac{dy}{dt}+m_2\frac{dy'}{dt}}{m+m_1+m_2}.$$

Multiply (2) by $2m_2\frac{dx'}{dt}-2m_2\frac{m_2\frac{dx'}{dt}+m_1\frac{dx}{dt}}{m+m_1+m_2}$,

and $2m_2\frac{dy'}{dt}-2m_2\frac{m_2\frac{dy'}{dt}+m_1\frac{dy}{dt}}{m+m_1+m_2}$.

Adding the four products we get

$$2m_1\left(\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}+\frac{dy}{dt}\cdot\frac{d^2y}{dt^2}\right)+2m_2\left(\frac{dx'}{dt}\cdot\frac{d^2x'}{dt^2}+\frac{dy'}{dt}\cdot\frac{d^2y'}{dt^2}\right)$$

$$-2\frac{m_1\frac{dx}{dt}+m_2\frac{dx'}{dt}}{m+m_1+m_2}\left(m_1\frac{d^2x}{dt^2}+m_2\frac{d^2x'}{dt^2}\right)$$

$$-2\frac{m_1\frac{dy}{dt}+m_2\frac{dy'}{dt}}{m+m_1+m_2}\left(m_1\frac{d^2y}{dt^2}+m_2\frac{d^2y'}{dt^2}\right)$$

$$-2m\left(\frac{m_1}{r^2}\frac{dr}{dt}+\frac{m_2}{r_1^2}\frac{dr_1}{dt}\right)-2\frac{d}{dt}\left(\frac{m_1m_2}{\sqrt{[(x'-x)^2+(y'-y)^2]}}\right)=0.$$

Integrating we get

$$m_1[(dx/dt)^2+(dy/dt)^2]+m_2[(dx'/dt)^2+(dy'/dt)^2]$$

$$-\frac{[(m_1dx/dt+m_2dx'/dt)^2+(m_1dy/dt+m_2dy'/dt)^2]}{m+m_1+m_2}$$

$$-2m\left(\frac{m_1}{r}+\frac{m_2}{r_1}\right)-\frac{2m_1m_2}{\sqrt{[(x'-x)^2+(y'-y)^2]}}=A,$$

or $mm_1[(dx/dt)^2+(dy/dt)^2]+mm_2[(dx'/dt)^2+(dy'/dt)^2]$

$$+m_1m_2[(dx'/dt-dxdt)^2+(dy'/dt-dy/dt)^2]$$

$$-2(m+m_1+m_2)\left[m\left(\frac{m_1}{r}+\frac{m_2}{r_1}\right)+\frac{m_1m_2}{\sqrt{[(x'-x)^2+(y'-y)^2]}}\right]=A.$$

The *vis viva* equation of motion.

DIOPHANTINE ANALYSIS.

82. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

In the series $1^3+3^3+5^3\ldots.$ find n so that the nth term and the sum of n terms shall both be squares.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The conditions of the problem require $1^3+3^3+5^3+\ldots.+(2n-1)^3 = n^2(2n^2-1)=\square$, and $(2n-1)^3=\square$.

Whence it follows that $2n-1=\square$, and $2n^2-1=\square$.

$2n-1=\square$, when $n=r^2+(r-1)^2$, as 1, 5, 13, 25, 41, etc.

Hence $2n^2-1=2[r^2+(r-1)^2]^2-1=\square$.

Whence $2[r^2+(r-1)^2]^2=\square+1$, two times a square equals the sum of two squares, the general formula for which is $2(p^2+q^2)^2=[(p+q)^2-2q^2]^2 +[(p-q)^2-2q^2]^2$.

Then $p^2+q^2=r^2+(r-1)^2\ldots.(1)$, and $(p-q)^2-2q^2=\pm1\ldots.(2)$.

From (1), put $p=r$ and $q=(r-1)$; and substituting in (2) we obtain $1-2(r-1)^2=\pm1$, or $2(r-1)^2=0$ or 2.

Whence $r-1=0$ or ±1, and $r=1$ or 2 or 0.

Substituting these values of r in $n=r^2+(r-1)^2$, we find $n=1$ and 5, apparently the only integral values.

When $n=1$, $2n-1=1$, $n^2(2n^2-1)=1$, and $(2n-1)^3=1$.

When $n=5$, $2n-1=9=3^2$, $n^2(2n^2-1)=1^3+3^3+5^3+7^3+9^3=35^2$, and $(2n-1)^3=(3^2)^3=(3^3)^2=27^2$.

Also solved by *G. B. M. ZERR, J. SCHEFFER, COOPER D. SCHMITT*, and the *PROPOSER*.

AVERAGE AND PROBABILITY.

88. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Find the average volume of the tetrahedron formed by joining four random points in a sphere.

Solution by the PROPOSER.

Let GK be the diameter of the section of the sphere made by a plane through the three random points A, B, C; HL the diameter of a parallel section through the fourth random point D; M, N the centers of these sections, respectively; O the center of the sphere; OP a line such that AB is always parallel to the plane MOP.

Let $OG=OH=r$, $MA=x$, $AB=y$, $AC=z$, $DN=u$, $\angle GOM=\theta$, $\angle BAM=\varphi$, $\angle CAM=\psi$, $\angle HON=\beta$, $\angle MOP=\lambda$, and the angle the plane POM makes with a fixed plane through $OP=\rho$.

An element of the sphere at A is $r\sin\theta d\theta 2\pi x dx$; at B, $y^2 dy d\varphi d\lambda$; at C, $\sin(\varphi+\psi)\sin\lambda z^2 dz d\psi d\rho$; at D, $r\sin\beta d\beta 2\pi u du$.

The limits of θ are 0 and π; of β, 0 and θ; of x, 0 and $r\sin\theta=x'$ and tripled; of u, 0 and $r\sin\beta=u'$ and sextupled; of φ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\varphi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and π; of y, 0 and $2x\cos\varphi=y'$; of z, 0 and $2x\cos\psi=z'$.

The area of the triangle $ABC=\frac{1}{2}yz\sin(\varphi+\psi)$.

Altitude of tetrahedron$=r(\cos\beta-\cos\theta)$.

Volume of tetrahedron$=\frac{1}{6}ryz\sin(\varphi+\psi)(\cos\beta-\cos\theta)$.

Since the whole number of ways four points can be taken is $(\frac{4}{3}\pi r^3)^4$, the required average triangle is

$$\triangle = \frac{729}{128\pi^4 r^{12}}\int_0^{\pi}\int_0^{\theta}\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{y'}\int_0^{z'} \tfrac{1}{6}ryz\sin(\varphi+\psi)$$

$$\times(\cos\beta-\cos\theta)r\sin\theta d\theta 2\pi x dx r\sin\beta d\beta 2\pi u du \sin(\varphi+\psi)d\varphi d\psi\sin\lambda d\lambda d\rho y^2 dy z^2 dz$$

$$= \frac{243}{16\pi^2 r^9}\int_0^{\pi}\int_0^{\theta}\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{y'} \sin\theta x^5\sin\beta u\sin^2(\varphi+\psi)\cos^4\psi$$

$$\times\sin\lambda(\cos\beta-\cos\theta)d\theta d\beta dx du d\varphi d\psi d\lambda d\rho y^3 dy$$

$$= \frac{243}{4\pi^2 r^9}\int_0^{\pi}\int_0^{\theta}\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi} \sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u\sin^2$$

$$\times(\varphi+\psi)\cos^4\varphi\cos^4\psi\sin\lambda d\theta d\beta dx du d\varphi d\psi d\lambda d\rho$$

$$= \frac{243}{2\pi r^9}\int_0^{\pi}\int_0^{\theta}\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi} \sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u\sin^2(\varphi+\psi)$$

$$\times\cos^4\varphi\cos^4\psi d\theta d\beta dx du d\varphi d\psi$$

$$= \frac{81}{64\pi r^9}\int_0^{\pi}\int_0^{\theta}\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u(15\pi\cos^4\varphi$$

$$-12\pi\cos^6\varphi+30\varphi\cos^4\varphi-24\varphi\cos^6\varphi-4\sin\varphi\cos^7\varphi+30\sin\varphi\cos^5\varphi)d\theta d\beta dx du d\varphi$$

$$= \frac{1215\pi}{256\, r^9}\int_0^{\pi}\int_0^{\theta}\int_0^{x'}\int_0^{u'} \sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u d\theta d\beta dx du$$

$$= \frac{1215\pi}{512\, r^7}\int_0^{\pi}\int_0^{\theta}\int_0^{x'} \sin\theta\sin^3\beta(\cos\beta-\cos\theta)x^9 d\theta d\beta dx$$

$$= \frac{243\pi r^3}{1024}\int_0^{\pi}\int_0^{\theta} \sin^{11}\theta\sin^3\beta(\cos\beta-\cos\theta)d\theta d\beta$$

$$= \frac{81\pi r^3}{4096}\int_0^{\pi} \sin^{11}\theta(8-4\sin^2\theta-\sin^4\theta-8\cos\theta)d\theta = \frac{36\pi r^3}{715}.$$

MISCELLANEOUS.

81. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

A cask in the form of a middle frustum of a spheroid, middle diameter $2b$, end diameters each $2c$, length $2d$, is lying in a horizontal position. The distance from middle of top to water is $b+e$, $e<b-c$. How much water is in the cask?

Solution by the PROPOSER.

Let $x^2/a^2+(y^2+z^2)/b^2=1$, be the equation to the spheroid.

Then $a^2=b^2d^2/(b^2-c^2)$, $z=(1/a)\sqrt{[b^2(a^2-x^2)-a^2y^2]}$.

$$\therefore\ V=(4/a)\int_0^d\int_e^{(b/a)\sqrt{(a^2-x^2)}}\sqrt{[b^2(a^2-x^2)-a^2y^2]}\,dx\,dy$$

$$=\frac{\pi b^2}{a^2}\int_0^d(a^2-x^2)dx-(2e/a)\int_0^d\sqrt{[a^2(b^2-e^2)-b^2x^2}\,dx$$

$$-\frac{2b^2}{a^2}\int_0^d(a^2-x^2)\sin^{-1}\left(\frac{ae}{b\sqrt{(a^2-x^2)}}\right)dx=\frac{\pi b^2}{3a^2}(3a^2d-d^3)$$

$$-\frac{de}{a}\sqrt{[a^2(b^2-e^2)-b^2d^2]}-\frac{ae(b^2-e^2)}{b}\sin^{-1}\left(\frac{bd}{a\sqrt{(b^2-e^2)}}\right)$$

$$-\frac{2b^2d}{3a^2}(3a^2-d^2)\sin^{-1}\left(\frac{ae}{b\sqrt{(a^2-d^2)}}\right)+\frac{2b^2e}{3a}\int_0^d\frac{x^2(3a^2-x^2)dx}{(a^2-x^2)\sqrt{[a^2(b^2-e^2)-b^2x^2}}$$

$$D=\frac{2b^2e}{3a}\int_0^d\frac{x^2(3a^2-x^2)dx}{(a^2-x^2)\sqrt{[a^2(b^2-e^2)-b^2x^2}}=\frac{2b^2e}{3a}\int_0^d\frac{x^2dx}{\sqrt{[a^2(b^2-e^2)-b^2x^2}}$$

$$+\frac{4ab^2e}{3}\int_0^d\frac{x^2dx}{(a^2-x^2)\sqrt{[a^2(b^2-e^2)-b^2x^2}}.$$

Let $bx=a\sqrt{(b^2-e^2)}\sin\theta$, $\theta_1=\sin^{-1}\{bd/[a\sqrt{(b^2-e^2)}]\}$.

$$\therefore\ D=\frac{2ae(b^2-e^2)}{3b}\int_0^{\theta_1}\sin^2\theta d\theta+\frac{4abe(b^2-e^2)}{3}\int_0^{\theta_1}\frac{\sin^2\theta d\theta}{b^2-(b^2-e^2)\sin^2\theta}$$

$$=\frac{4ab^2}{3}\tan^{-1}\left(\frac{de}{\sqrt{[a^2(b^2-e^2)-b^2d^2]}}\right)-\frac{ae(e^2+3b^2)}{3b}\sin^{-1}\left(\frac{bd}{a\sqrt{(b^2-e^2)}}\right)$$

$$-(de/3a)\sqrt{[a^2(b^2-e^2)-b^2d^2]}.$$

$$\therefore\ V=\frac{b^2d}{3a^2}(3a^2-d^2)\left[\pi-2\sin^{-1}\left(\frac{ae}{b\sqrt{(a^2-d^2)}}\right)\right]-\frac{4de}{3a}\sqrt{[a^2(b^2-e^2)-b^2d^2]}$$

$$+\frac{4ab^2}{3}\tan^{-1}\left(\frac{de}{\sqrt{[a^2(b^2-e^2)-b^2d^2]}}\right)+\frac{2ae}{3b}(e^2-3b^2)\sin^{-1}\left(\frac{bd}{a\sqrt{(b^2-e^2)}}\right).$$

But $a=bd/\sqrt{(b^2-c^2)}$.

$$\therefore V=\frac{d(2b^2+c^2)}{3}[\pi-2\sin^{-1}(e/c)]-(4de/3)\sqrt{(c^2-e^2)}$$

$$+\frac{2d}{3\sqrt{(b^2-c^2)}}\left\{2b^3\sin^{-1}\left[e/c\sqrt{\left(\frac{b^2-c^2}{b^2-e^2}\right)}\right]+e(e^2-3b^2)\sin^{-1}\sqrt{\left(\frac{b^2-c^2}{b^2-e^2}\right)}\right\}.$$

A good solution, with diagram, was received from *J. SCHEFFER*.

BIOGRAPHICAL SKETCH OF SYLVESTER ROBINS.

Sylvester Robins was born in Union Township, Unterdon County, New Jersey, December 14, 1834, and died at his home in North Branch, New Jersey, April 25, 1900, in the 66th year of his age.

During early boyhood, he attended the common school in the neighborhood, and very early manifested a deep interest in his studies.

At the age of 12 he entered the preparatory school of Rev. John Derveer, D. D., at Easton, Pennsylvania, where he continued for six years, during the last two years of which he was employed as assistant teacher. During the years from 1852 to 1873, Mr. Robins was engaged as teacher at Cedar Grove, Readington, and Bloomsberg, New Jersey, and for six and a half years at Easton, Pennsylvania.

In 1858 Mr. Robins was married to Miss Sarah J. Bird, and of this marriage there were born seven children—one of whom is Edward R. Robins, Instructor in Mathematics in Albany Academy, Albany, New York, and who is the author of *Algebra Reviews*, a small book published by Ginn & Co. before its author had reached the age of 21.

Mr. Robins loved mathematics and lamented the fact that his early opportunities had been so limited, and delighted to revel in finding series of rational triangles, rational trapezoids, or rational parallelopipeds, for which he found "keys." Numerous problems of this character are proposed by him in the MONTHLY, and in the *Mathematical Magazine* and *Mathematical Visitor* published by Dr. Artimas Martin. It was somewhat surprising, how, in continued fractions from different numbers he would see the "key" to some kind of a mathematical figure. In his last letter to Mr. C. A. Roberts, an intimate friend of his and a mathematician interested in very much the same line of investigation, Mr. Robins wrote, "I hope you will indulge the spirit in which I write and pardon the writer, who can no more help his desire to ask you for aid, than he can escape the sight of triangles in the leaves, or of parallelopipeds in the bricks under his feet."

Mr. Robins was a contributor to the MONTHLY from the first, and there are yet a number of his contributions in his favorite line of work in the hands of the editor.

BOOKS.

Plane and Solid Geometry.—Inductive Method. By A. A. Dodd, M. S. D., S. B., and B. F. Chace, Teachers of Mathematics in the Manual Training High School, Kansas City, Mo. 8vo. Cloth, 406 pages. Price, $1.00. Kansas City: Hudson-Kimberly Publishing Co.

This work presupposes, on the part of the student, a knowledge of Inventional or Constructional Geometry. The purpose of the work is good, and in the hands of a skillful and well trained teacher the results from the study of such a book would be very good; but in the hands of a teacher whose knowledge of geometry is somewhat deficient, its study would certainly be unsatisfactory. The authors have the right view of presenting the subject, and are to be congratulated in the courage they have manifested in presenting the work for public recognition. B. F. F.

A Brief History of Mathematics. An Authorized Translation of Dr. Karl Fink's Geschichte der Elementar-Mathematik. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Principal of the State Normal School at Brockport, New York. 8vo, Red Cloth, 333 pages. Price $1.50. Chicago: The Open Court Publishing Co.

This work briefly states the facts of mathematical history. It is not a book of anecdotes, nor one of biography. The author systematically traces the development of the science of mathematics from the earliest times down to the present. Geometry is reviewed from the primitive ideas of the Babylonians to the projective and differential geometry and the science of n-dimensional space and trigonometry from the ideas of Ahmes to the refined notions of recent times. B. F. F.

Elements of Algebra. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Principal of the State Normal School at Brockport, New York.

In this text-book, the authors have followed their usual plan of allowing the light of modern mathematics to shine in upon the old. "And this is the condemnation, that light has come into the world and men love darkness rather than light." An examination of this book will commend it to all good teachers of algebra. B. F. F.

Bestimmung der Coefficienten welche bei der Berechnung der Intergale

$$\int \frac{x^n dx}{\sqrt{1+ax+bx^2}} \text{ und } \int \frac{x^n dx}{\sqrt{1+ax^2+bx^2+cx^3}}$$

auftreten. Von Henry Benner. Pamphlet, 60 pages. Chicago: Ginn & Co.

This dissertation was offered by its author as a partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University at Erlangen, Germany. B. F. F.

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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

(Continued from the June-July Number.)

CHAPTER VII.

GEOMETRICAL MULTIPLICATION.

88. Basing our investigation on the fundamental law of combinatory multiplication (34), *let us seek the product of a non-posited point* (76) *and two vectors.* The vectors are thought of as denoting merely translation a given distance in a given direction (See 4—9). Let p denote the point and α and β the vectors.

Suppose

$$\alpha = x_1\varepsilon_1 + y_1\varepsilon_2$$
$$\beta = x_2\varepsilon_1 + y_2\varepsilon_2$$

where ε_1 and ε_2 are unit vectors at right angles to each other. Then by 45

$$[p\alpha\beta] = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} [p\varepsilon_1\varepsilon_2].$$

Now the determinant $x_1y_2 - x_2y_1$ is the difference between two rectangles.